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Expansions of Algebras and Superalgebras and Some Applications

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Abstract After reviewing the three well-known methods to obtain Lie algebras and superalgebras from given ones, namely, contractions, deformations and extensions, we describe a fourth method recently introduced, the expansion of Lie (super)algebras. Expanded (super)algebras have, in general, larger dimensions than the original algebra, but also include the İnönü–Wigner and generalized IW contractions as a particular case. As an example of a physical application of expansions, we discuss the relation between the possible underlying gauge symmetry of eleven-dimensional supergravity and the superalgebra $osp(1|32)$.

1 Introduction

Different constructions describing the symmetry of physical theories have made their way into physics, gradually avoiding previously established 'no-go theorems'. That was, for instance, the case of Lie superalgebras, nowadays ubiquitous in theoretical physics and

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brought into the picture as a way of mixing spacetime and internal symmetries, not allowed in a purely bosonic context (see [1]). This led to the advent of supersymmetry, a symmetry which involves bosons and fermions simultaneously. From a mathematical point of view, and setting aside its Lorentz part, the super-Poincaré algebra is a central extension of an odd (fermionic, spinorial) Abelian algebra by the spacetime translations (see [2, 3]) but it is not, however, the most general spacetime superalgebra. In fact, larger supersymmetry algebras going beyond the restrictions of the Haag–Łopuszański–Sohnius theorem [4] have appeared in connection with the description of different supersymmetric theories. For instance, the quasi-invariance under standard supersymmetry of the Wess–Zumino (WZ) terms of the super-p-brane actions results in algebras realized by the conserved supercharges that include additional (topological) charges [5] and that are extensions of the original, $\{Q, Q\} \sim P$ minimal supersymmetry algebra. Also—an example to be discussed in Sect. 4—an $osp(1|32)$-related gauge formulation of $D = 11$ supergravity [6] requires a gauge algebra that includes an additional fermionic generator [7]. It thus makes sense to use supersymmetry algebras beyond the standard super-Poincaré algebra, and many have been introduced in various contexts, leading also to a variety of generalized, enlarged superspaces (see [5, 7–16] and references therein).

New algebras and superalgebras may be related to, or derived from, previously known ones. With this in mind, we first comment on the three well known ways to obtain new (super)algebras from given ones, *i.e.* contractions, deformations and extensions of Lie and super Lie algebras. Then, we describe in Sect. 3 a new procedure [17, 18] (see also [19]), that *includes* the İnönü–Wigner (IW) and generalized contractions, the method of Lie (super)algebra expansions, which makes use of the geometrical structure of the algebra as expressed by the Maurer–Cartan one-forms. At the end of Sect. 3 the very recent method of S-expansions of Lie (super)algebras [20] is also briefly described. We conclude with an application in Sect. 4, where we show how our expanded algebras appear [21, 22] in the discussion of the relation between $OSp(1|32)$ and the possible underlying gauge symmetry group of $D = 11$ supergravity [6].

2 Lie Algebras and Superalgebras from Given Ones

Let $\mathcal{G}$ be a finite-dimensional Lie (super)algebra with basis $\{X_i\}$, which may be realized by left-invariant (LI) generators $X_i(g)$ on the corresponding (super)group manifold G with local coordinates g^i, $i = 1, \ldots, \dim G = \dim \mathcal{G}$. Let c_{ij}^k be the structure constants of $\mathcal{G}$ in the basis $\{X_i\}$, $[X_i, X_j] = c_{ij}^k X_k$. Let $\{\omega^i(g)\}$, $i = 1, \ldots, \dim G$, be the basis determined by the (dual, LI) Maurer–Cartan (MC) one-forms on G. The MC equations that characterize $\mathcal{G}$, in a way dual to its Lie bracket description, are given by

$$d\omega^k(g) = -\frac{1}{2} c_{ij}^k \omega^i(g) \wedge \omega^j(g), \quad i, j, k = 1, \ldots, \dim \mathcal{G}. \tag{2.1}$$

The standard procedures to obtain new (super)algebras from given ones are:

(a) *Contractions*

Contractions go back to the work of Segal, Inönü and Wigner (see [23–26]). In its simplest İnönü–Wigner (IW) form [24, 25], the contraction of $\mathcal{G}$ with respect to a subalgebra $\mathcal{L}_0 \subset \mathcal{G}$ is performed by rescaling the generators of the coset $\mathcal{G}/\mathcal{L}_0$, and then by taking a singular limit for the rescaling parameter. This procedure may be extended to generalized

IW contractions in the sense of Weimar-Woods (W-W) [27, 28]. These are defined when the vector space W of $\mathcal{G}$ can be split as a sum of $n+1$ subspaces

$$\mathcal{G}: W = V_0 \oplus V_1 \oplus \cdots \oplus V_n = \oplus_s V_s, \quad s = 0, 1, \ldots, n, \tag{2.2}$$

such that the following W-W conditions are satisfied:

$$c_{i_p j_q}^{k_s} = 0 \quad \text{if } s > p+q \quad \text{i.e.} \quad [V_p, V_q] \subset \oplus_{s \le p+q} V_s, \quad p, q = 0, 1, \ldots, n, \tag{2.3}$$

where $i_p = 1, \ldots, \dim V_p$ labels the generators of $\mathcal{G}$ in V_p (we have written above $s \le p+q$ rather than $s \le \min\{p+q, n\}$ for simplicity.) Clearly, condition (2.3) implies that V_0 is a subalgebra $\mathcal{L}_0$ of $\mathcal{G}$. The contracted algebra $\mathcal{G}_c$ is obtained after the generators of each subspace are properly re-scaled [27, 28] and a singular limit for the scaling parameter λ is taken. $\mathcal{G}_c$ has the same dimension as $\mathcal{G}$; the case $n = 1$ reproduces the simple IW contraction. There have been other variations of the IW contraction procedure (see *e.g.* [29–36]); in particular, the 'graded contractions' [34, 35] may be expressed as generalized IW ones (see [27, 28] and the contribution of E. Weimar-Woods to these proceedings). All contractions have in common that $\mathcal{G}$ and $\mathcal{G}_c$ have, necessarily, the same dimension as vector spaces.

Well known examples of contractions relevant in physics include the Galilei algebra as an IW contraction of the Poincaré algebra, the Poincaré algebra as a contraction of the de Sitter algebras [37], or the characterization of the M-theory superalgebra [9, 10] as a contraction (*ignoring* the Lorentz part, *cf.* [18]) of $osp(1|32)$.

The contraction process has also been considered for 'quantum' algebras (see *e.g.*, [38, 39]) and used, in particular, to obtain the κ-Poincaré [40] and κ-Galilei algebras [41].

(b) *Deformations*

From a physical point of view, Lie algebra deformations [42–45] can be regarded as a process inverse to contractions (see also [27, 28, 37, 46–48]). Mathematically, a deformation $\mathcal{G}_d$ of a Lie algebra $\mathcal{G}$ is a Lie algebra 'close', but not isomorphic, to $\mathcal{G}$. As in the case of $\mathcal{G}_c$ above, $\mathcal{G}_d$ has the same dimension as $\mathcal{G}$.

Deformations are obtained by modifying the *r.h.s.* of the original commutators by adding new terms that depend on a parameter t in the form

$$[X, Y]_t = [X, Y]_0 + \sum_{i=1}^{\infty} \omega_i(X, Y) t^i, \quad X, Y \in \mathcal{G}, \ \omega_i(X, Y) \in \mathcal{G}. \tag{2.4}$$

Checking the Jacobi identities up to $O(t^2)$, it is seen that the expression satisfied by ω_1 characterizes it as a two-cocycle. Thus, the second Lie algebra cohomology group $H^2(\mathcal{G}, \mathcal{G})$ of $\mathcal{G}$ with coefficients in the Lie algebra $\mathcal{G}$ itself is the group of infinitesimal deformations of $\mathcal{G}$ and $H^2(\mathcal{G}, \mathcal{G}) = 0$ is a *sufficient* condition for rigidity [42, 43, 45]. In this case, $\mathcal{G}$ is *rigid* or *stable* under infinitesimal deformations; any attempt to deform it yields an isomorphic algebra. The problem of finite deformations depends on the integrability of the infinitesimal deformation; the obstruction is governed by the third cohomology group $H^3(\mathcal{G}, \mathcal{G})$, which needs being trivial.

As is well known, the Poincaré algebra may be seen as a deformation of the Galilei one, a fact that may be viewed as a group theoretical prediction of relativity. The de Sitter, $so(4, 1)$, and anti de Sitter, $so(3, 2)$, algebras are stabilizations of the Poincaré algebra; $osp(1|4)$ is a deformation of the $N = 1$, $D = 4$ super-Poincaré algebra [49]. Quantization itself may also be looked at as a deformation (see [50–54]), the classical limit being the contraction limit

$\hbar \to 0$. Nontrivial central extensions of Lie algebras may also be considered as deformations or partial stabilizations of trivial (direct sum) extensions.

(c) *Extensions (of a Lie algebra or superalgebra by another one)*

In contrast with the procedures (a) and (b) above, the initial data of the extension problem includes *two* Lie algebras $\mathcal{G}$ and $\mathcal{A}$. A Lie algebra $\tilde{\mathcal{G}}$ is an extension of $\mathcal{G}$ by $\mathcal{A}$ if $\mathcal{A}$ is an ideal of $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{G}}/\mathcal{A} = \mathcal{G}$. As a result, $\dim \tilde{\mathcal{G}} = \dim \mathcal{G} + \dim \mathcal{A}$, so that the extension process is also 'dimension preserving'. To obtain an extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by $\mathcal{A}$ it is necessary to specify first an action ρ of $\mathcal{G}$ on $\mathcal{A}$ *i.e.*, a Lie algebra homomorphism $\rho : \mathcal{G} \to \mathrm{End}\,\mathcal{A}$. The possible extensions $\tilde{\mathcal{G}}$ for a given set $(\mathcal{G}, \mathcal{A}, \rho)$ and the possible obstructions to the extension process are, again, governed by cohomology (see [3] for full details and references).

Examples of extensions in physics are the centrally extended Galilei algebra, which is relevant in non-relativistic quantum mechanics (and that may be obtained as a contraction of the trivially extended $D = 4$ Poincaré group, see [55] to see how contractions may generate cohomology), the two-dimensional extended Poincaré algebra that allows [56, 57] for a gauge theoretical derivation of the Callan–Giddings–Harvey–Strominger model [58] for two-dimensional gravity, or the M-theory superalgebra that, without its Lorentz automorphisms part, is the maximal central extension of the Abelian $D = 11$ supertranslations algebra ([5, 8–10, 13, 14]).

We now turn to a new procedure, the expansion of Lie algebras and superalgebras.

3 Expansions of Lie (Super)Algebras

Under a different name, Lie algebra expansions were first used in [17], and then the method was studied in general in [18] (see also [19]). The idea is to perform a rescaling by a parameter λ of some of the group coordinates g^i, $i = 1, \ldots, \dim \mathcal{G}$. Consequently, the MC one-forms $\omega^i(g, \lambda)$ of $\mathcal{G}$ are expanded as power series in λ. Inserting these expansions (polynomials in λ) in the original MC equations for $\mathcal{G}$, one obtains a set of equations that have to be satisfied, each one corresponding to a power of λ. The problem at this stage is how to cut the series expansions of the different ω^i's in such a way that the resulting set of MC-like equations be closed under d, so that it defines the MC equations of a new, finite-dimensional *expanded* Lie algebra.

In fact, notice that it is possible to write the MC forms $\omega^i(g)$ of $\mathcal{G}$ as polynomials in the group coordinates g^i (see [18]) as

$$\omega^i(g) = \left[\delta^i_j + \frac{1}{2!} c^i_{jk} g^k + \sum_{n=2}^{\infty} \frac{1}{(n+1)!} c^{h_1}_{jk_1} c^{h_2}_{h_1 k_2} \ldots c^{h_{n-1}}_{h_{n-2} k_{n-1}} c^i_{h_{n-1} k_n} g^{k_1} g^{k_2} \ldots g^{k_{n-1}} g^{k_n} \right] dg^j . \tag{3.1}$$

Hence, a redefinition

$$g^l \to \lambda^{q_l} g^l \tag{3.2}$$

of *some* group coordinates g^l will produce an expansion of the MC one-forms $\omega^i(g, \lambda)$ as a sum of one-forms $\omega^{i,\alpha}(g)$ on G multiplied by the corresponding powers λ^α of λ. The actual form of the power series of $\omega^i(g, \lambda)$ is in fact dependent on the possible structure of $\mathcal{G}$ if a suitable redefinition of the group parameters is made. In general, moreover, the richer the structure of $\mathcal{G}$, the more possibilities arise to cut the ω^i's power series in order to obtain well defined finite-dimensional Lie algebras.

For the sake of definiteness, let us discuss the case in which $\mathcal{G}$ satisfies the Weimar-Woods (W-W) conditions (2.2), (2.3), referring to [18] for other interesting cases. When the W-W conditions are satisfied, the MC one-forms of $\mathcal{G}$ arrange themselves in $n+1$ sets $\{\omega^{i_p}\}$, $i_p = 1, \dots, \dim V_p$, $p = 0, 1, \dots, n$, corresponding to each subspace V_p in (2.2), and the structure constants of $\mathcal{G}$ satisfy $c_{i_p j_q}^{k_s} = 0$ if $s > p+q$; the subspace V_0 is a subalgebra $\mathcal{L}_0$ of $\mathcal{G}$. Consider next the rescaling $g^{i_p} \to \lambda^p g^{i_p}$, $p = 0, \dots, n$, or, explicitly,

$$g^{i_0} \to g^{i_0}, \qquad g^{i_1} \to \lambda g^{i_1}, \qquad \dots, \qquad g^{i_n} \to \lambda^n g^{i_n}, \tag{3.3}$$

of the group parameters, where g^{i_p} is subordinated to the splitting (2.2) in an obvious way. With this rescaling, the condition (2.3), namely, $c_{i_p j_q}^{k_s} = 0$ if $s > p+q$, produces that the series expansion of the forms ω^{i_p} in each subspace V_p that results from the insertion of (3.3) in (3.1), starts with the power λ^p, $p = 0, 1, \dots, n$ [18]:

$$\omega^{i_p} = \sum_{\alpha_p = p}^{\infty} \omega^{i_p, \alpha_p} \lambda^{\alpha_p} = \lambda^p \omega^{i_p, p} + \lambda^{p+1} \omega^{i_p, p+1} + \cdots, \tag{3.4}$$

where the index denoting each power of λ has been written as α_p to stress the fact that the series expansion is different for each ω^{i_p}, $p = 0, 1, \dots, n$.

Inserting the series (3.4) into the MC equations (2.1) of $\mathcal{G}$ and equating the coefficients with the same powers of λ, a set of equations for the various coefficient one-forms ω^{i_p, α_p} is obtained:

$$d\omega^{k_s, \alpha_s} = -\frac{1}{2} C_{i_p, \beta_p\, j_q, \gamma_q}^{k_s, \alpha_s} \omega^{i_p, \beta_p} \wedge \omega^{j_q, \gamma_q}, \tag{3.5}$$

where

$$C_{i_p, \beta_p\, j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s, \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s, \end{cases} \qquad \begin{array}{l} p, q, s = 0, 1, \dots, n, \\ i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s}, \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \dots, N_p \end{array} \tag{3.6}$$

and the $c_{i_p j_q}^{k_s}$ satisfy (2.3).

One may now consider whether the series (3.4) for each ω^{i_p} may be cut at an arbitrary order N_p *i.e.*, whether any finite number of one-form coefficients ω^{i_p, α_p}, $\alpha_p = p, p+1, \dots, N_p$, can be retained in such a way that equations (3.5), (3.6) define, respectively, the MC equations and structure constants of a new, finite-dimensional Lie algebra labeled $\mathcal{G}(N_0, \dots, N_n)$. This is clearly not the case. For $\mathcal{G}(N_0, \dots, N_n)$ to be a Lie algebra, two conditions must be met:

(a) the set of retained one-form coefficients,

$$\{\omega^{i_0, 0}, \omega^{i_0, 1}, \overset{N_0+1}{\dots}, \omega^{i_0, N_0}; \omega^{i_1, 1}, \overset{N_1}{\dots}, \omega^{i_1, N_1}; \dots; \omega^{i_n, n}, \overset{N_n - n+1}{\dots}, \omega^{i_n, N_n}\}, \tag{3.7}$$

which determines the dimension of the expanded algebra $\mathcal{G}(N_0, \dots, N_n)$ by

$$\dim \mathcal{G}(N_0, \dots, N_n) = \sum_{p=0}^{n} (N_p - p + 1) \dim V_p, \tag{3.8}$$

must be closed under the exterior differential d; and

(b) the symbols $C^{k_s,\alpha_s}_{i_p,\beta_p j_q,\gamma_q}$ defined in (3.6) must obey the Jacobi identity (notice that their definition (3.6) makes them already inherit the symmetry properties of the structure constants $c^{k_s}_{i_p j_q}$ of the original (super)algebra).

With regard to the condition a) notice that, due to the W-W conditions (2.3), the forms ω^{i_p,β_p} that enter the expression of $d\omega^{k_s,\alpha_s}$ in (3.5) are those with [18]

$$\beta_p \leq \begin{cases} \alpha_s - s + p, & \text{if } p \leq s, \quad p, s = 0, 1, \dots, n, \\ \alpha_s, & \text{if } p > s, \quad \alpha_p, \beta_p = p, p+1, \dots, N_p. \end{cases} \tag{3.9}$$

Hence, the set of forms (3.7) will be closed under d if the cutting orders satisfy

$$N_p \geq \begin{cases} N_s - s + p, & \text{if } p \leq s, \quad p, s = 0, 1, \dots, n, \\ N_s, & \text{if } p > s, \quad \alpha_p, \beta_p = p, p+1, \dots, N_p, \end{cases} \tag{3.10}$$

namely, when

$$N_{p+1} = N_p \quad \text{or} \quad N_{p+1} = N_p + 1 \quad (p = 0, 1, \dots, n-1), \tag{3.11}$$

which gives [18] 2^n possibilities in all.

As for the condition (b), the Jacobi identities

$$\begin{aligned} C^{k_s,\alpha_s}_{i_p,\beta_p[j_q,\gamma_q} C^{i_p,\beta_p}_{l_t,\rho_t m_u,\sigma_u]} &= 0 \\ &= C^{k_s,\alpha_s}_{i_p,\beta_p j_q,\gamma_q} C^{i_p,\beta_p}_{l_t,\rho_t m_u,\sigma_u} + C^{k_s,\alpha_s}_{i_p,\beta_p m_u,\sigma_u} C^{i_p,\beta_p}_{j_q,\gamma_q l_t,\rho_t} \\ &\quad + C^{k_s,\alpha_s}_{i_p,\beta_p l_t,\rho_t} C^{i_p,\beta_p}_{m_u,\sigma_u j_q,\gamma_q}, \end{aligned} \tag{3.12}$$

are satisfied through those for $\mathcal{G}$. This is a consequence of the fact that, for $\mathcal{G}$, the exterior derivative of the λ-expansion of the MC equations is the λ-expansion of their exterior derivative, but it may also be seen directly.

Indeed, we only need to check that (3.12) reduces to the Jacobi identities for $\mathcal{G}$ when the order in the upper index is the sum of those in the lower ones since the C's are zero otherwise. First we see that, when $\alpha_s = \gamma_q + \rho_t + \sigma_u$, all three terms in the r.h.s. of (3.12) give non-zero contributions. This is so because the range of β_p is only limited by $\beta_p \leq \alpha_s$, which holds when $\beta_p = \rho_t + \sigma_u$, $\beta_p = \gamma_q + \rho_t$ and $\beta_p = \sigma_u + \gamma_q$. Secondly, and since $\beta_p \geq p$, we also need that the terms in the i_p sum that are suppressed in (3.12) when $p > \beta_p$ be also absent in the Jacobi identities for $\mathcal{G}$ so that (3.12) does reduce to the Jacobi identities for $\mathcal{G}$. Consider *e.g.*, the first term in the r.h.s. of (3.12). If $p > \beta_p$, then $p > \rho_t + \sigma_u$ and hence $p > t + u$. Thus, by the W-W condition (2.3), this term will not contribute to the Jacobi identities for $\mathcal{G}$ and no sum over the subspace V_p index i_p will be lost as a result. The argument also applies to the other two terms for their corresponding β_p's.

A particular solution to (3.11) is obtained by setting $N_p = p$, $p = 0, 1, \dots, n$, which defines $\mathcal{G}(0, 1, \dots, n)$, with $\dim \mathcal{G}(0, 1, \dots, n) = \dim \mathcal{G}$ by (3.8). Since in this case α_p takes only one value ($\alpha_p = N_p = p$) for each $p = 0, 1, \dots, n$, we may drop this label. Then, the structure constants (3.6) for $\mathcal{G}(0, 1, \dots, n)$ read

$$C^{k_s}_{i_p j_q} = \begin{cases} 0, & \text{if } p + q \neq s, \quad p = 0, 1, \dots, n, \\ c^{k_s}_{i_p j_q}, & \text{if } p + q = s, \quad i_{p,q,s} = 1, 2, \dots, \dim V_{p,q,s}, \end{cases} \tag{3.13}$$

which shows that $\mathcal{G}(0, 1, \ldots, n)$ is the generalized IW contraction of $\mathcal{G}$, in the sense of [27, 28], subordinated to the splitting (2.2). Obviously, if $n = 1$, $\mathcal{G} = \mathcal{L}_0 \oplus V_1$, where $\mathcal{L}_0$ is a subalgebra, and the simple IW contraction is recovered as the expansion $\mathcal{G}(0, 1)$.

Thus, we have actually proved the following

Theorem 1 *Let $\mathcal{G} = V_0 \oplus V_1 \oplus \cdots \oplus V_n$ be a splitting of $\mathcal{G}$ into $n + 1$ subspaces. Let $\mathcal{G}$ fulfill the Weimar-Woods contraction condition (2.3) subordinated to this splitting, $c_{i_p j_q}^{k_s} = 0$ if $s > p + q$. The one-form coefficients ω^{i_p, α_p} of (3.7) resulting from the expansion of the Maurer–Cartan forms ω^{i_p} in which $g^{i_p} \to \lambda^p g^{i_p}$, $p = 0, \ldots, n$ (see (3.3)), determine expanded Lie algebras, denoted $\mathcal{G}(N_0, N_1, \ldots, N_n)$, of dimension (3.8) and structure constants given by*

$$C_{i_p, \beta_p j_q, \gamma_q}^{k_s, \alpha_s} = \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s, \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s, \end{cases} \qquad \begin{array}{l} p, q, s = 0, 1, \ldots, n, \\ i_{p,q,s} = 1, 2, \ldots, \dim V_{p,q,s}, \\ \alpha_p, \beta_p, \gamma_p = p, p + 1, \ldots, N_p \end{array} \tag{3.14}$$

(see (3.6)) if $N_p = N_{p+1}$ or $N_p = N_{p+1} - 1$ $(p = 0, 1, \ldots, n - 1)$ in $(N_0, N_1, \ldots, N_n)$. In particular, the $N_p = p$ solution determines the algebra $\mathcal{G}(0, 1, \ldots, n)$, which is the generalized İnönü–Wigner contraction of $\mathcal{G}$.

In general, the Lie algebra $\mathcal{G}(N_0, N_1, \ldots, N_n)$ is larger than $\mathcal{G}$ (see (3.8)). This fact, and its derivation, justifies the name of *expanded* algebras [18].

An interesting case is that of Lie superalgebras, the splitting of which into subspaces naturally satisfies the W-W conditions. For instance, we may take $\mathcal{G} = V_0 \oplus V_1$ or $\mathcal{G} = V_0 \oplus V_1 \oplus V_2$ with V_0 or $V_0 \oplus V_2$ containing all the even (bosonic) generators and V_1 containing the Grassmann odd (fermionic) ones. Then, the expansions of the MC one-forms of V_1 (V_0 and V_2) only contain odd (even) powers of λ [18]. The consistency conditions for the existence of $\mathcal{G}(N_0, N_1)$-type expanded superalgebras require that

$$N_0 = N_1 - 1 \quad \text{or} \quad N_0 = N_1 + 1, \tag{3.15}$$

and, for the $\mathcal{G}(N_0, N_1, N_2)$ case, that one of the three following possibilities holds:

$$N_0 = N_1 + 1 = N_2, \qquad N_0 = N_1 - 1 = N_2, \qquad N_0 = N_1 - 1 = N_2 - 2. \tag{3.16}$$

This last case allows us to obtain, for example, the M-algebra including the Lorentz $SO(1, 10)$ automorphisms as the expansion $osp(1|32)(2, 1, 2)$ of $osp(1|32)$. The appropriate choice of V_0, V_1, V_2 leading to this expansion can be found in [18].

3.1 S-expansions of Lie (super)algebras

As we have seen, the expansion method allows us to obtain new Lie algebras of increasing dimensions from $\mathcal{G}$ by a geometric procedure based on expanding the MC forms. One may think of other possibilities leading, in general, to larger algebras from a given one. We conclude this section by briefly describing another construction, very recently proposed [20], which is based on combining the structure constants of $\mathcal{G}$ with the inner law of a semigroup S to define the Lie bracket of a new, S-expanded algebra. The ingredients here are, then, the algebra $\mathcal{G}$ and a certain semigroup S.

Consider a finite Abelian semigroup S (a set S with $\operatorname{ord} S$ elements $\alpha, \beta, \gamma, \ldots \in S$, endowed with a commutative and associative composition law $S \times S \to S$, $(\alpha, \beta) \mapsto \alpha\beta =$

$\beta\alpha$). Then, one may define a Lie algebra structure over the vector space obtained by taking ord S copies of $\mathcal{G}$,

$$\begin{aligned}&\mathcal{G}_S : W_\alpha \oplus W_\beta \oplus W_\gamma \oplus \cdots = \oplus_{\alpha\in S} W_\alpha \quad (W_\alpha \approx \mathcal{G}\ \forall\alpha),\\ &\dim \mathcal{G}_S = \operatorname{ord} S \times \dim \mathcal{G},\end{aligned} \tag{3.17}$$

by means of the structure constants

$$C_{i\alpha\ j\beta}^{k\gamma} = c_{ij}^{k}\delta_{\alpha\beta}^{\gamma}, \tag{3.18}$$

where δ is the Kronecker symbol and the subindex $\alpha\beta \in S$ denotes the inner composition in S so that $\delta_{\alpha\beta}^{\gamma} = 1$ when $\alpha\beta = \gamma$ in S and zero otherwise. The constants $C_{i\alpha j\beta}^{k\gamma}$ defined by (3.18) inherit the symmetry properties of the c_{ij}^{k} of $\mathcal{G}$ by virtue of the Abelian character of the S-product, and satisfy the Jacobi identity $C_{[i\alpha j\beta}^{h\delta} C_{k\gamma]h\delta}^{l\epsilon} = 0$ because of the commutativity and associativity of the semigroup inner law and the Jacobi identity of $\mathcal{G}$, $c_{[ij}^{h} c_{k]h}^{l} = 0$. This Lie (super)algebra $\mathcal{G}_S$ was called *S-expansion* of $\mathcal{G}$ [20].

When the Lie brackets of the original algebra $\mathcal{G}$ satisfy certain conditions, as *e.g.* the W-W conditions (2.3), then certain subalgebras $\mathcal{G}'_S$ can be extracted [20] from the S-expanded algebra $\mathcal{G}_S$ provided that it is possible to find subsets of S (see (3.22) below) the composition of which *mimics* the subspace structure of $\mathcal{G}$ with respect to its Lie bracket (see (2.3)). These $\mathcal{G}'_S$ can then be used to retrieve the expansions $\mathcal{G}(N_0, \ldots, N_n)$. The procedure is not entirely straightforward, so we shall make explicit the intermediate steps below.

Let then $\mathcal{G}$ satisfy the W-W conditions (2.3) and let us conveniently choose the semigroup S as [20]

$$S = \{\alpha \mid \alpha = 0, 1, \ldots, N, N+1\}, \qquad \alpha\beta = \begin{cases} \alpha+\beta, & \text{if } \alpha+\beta < N+1, \\ N+1, & \text{if } \alpha+\beta \geq N+1, \end{cases} \tag{3.19}$$

where $\alpha + \beta$ is simply the sum of natural numbers. The underlying vector space of any S-expanded algebra is $\mathcal{G}_S = \oplus_{\alpha\in S} W_\alpha$; each copy W_α of the vector space W of $\mathcal{G}$ obviously admits the same splitting, $W_\alpha = \oplus_p V_{p\alpha}$, $p = 0, \ldots, n$. Hence, the $\mathcal{G}_S$ vector subspace structure splits as

$$\mathcal{G}_S = \bigoplus_{p} \bigoplus_{\alpha\in S} V_{p\alpha}, \quad \text{where } V_{p\alpha} \approx V_p,\ p = 0, 1, \ldots, n,\ \alpha \in S. \tag{3.20}$$

As $\mathcal{G}$ satisfies the W-W conditions, $[V_p, V_q] \subset \bigoplus_{s\leq p+q} V_s$, $p, q = 0, 1, \ldots n$, the Lie bracket subspace structure of $\mathcal{G}_S$ inherited from that of $\mathcal{G}$ is

$$\mathcal{G}_S : [V_{p\alpha}, V_{q\beta}] \subset \bigoplus_{s\leq p+q} V_{s\alpha\beta}, \tag{3.21}$$

where $\alpha\beta$ in $V_{s\alpha\beta}$ again denotes S-composition (here again, and also below, we write $s \leq p + q$ rather than $s \leq \min\{p + q, n\}$ for simplicity's sake).

Let $\{S_s\}$ in

$$S = \bigcup_{s} S_s, \quad s = 0, 1, \ldots, n, \tag{3.22}$$

be a (not necessarily disjoint) collection of subsets $S_s \subset S$ (compare (3.22) and (2.2)). The subsets $S_s \subset S$ are thus in one-to-one correspondence with the vector subspaces $V_s \subset \mathcal{G}$ in (2.2). When the condition

$$S_p S_q \subset \bigcap_{s \le p+q} S_s, \qquad S_p S_q := \{\alpha_p \beta_q \mid \alpha_p \in S_p, \beta_q \in S_q\}, \tag{3.23}$$

is satisfied, the collection of subsets $S_s \subset S$ is adapted to the partition $V_s \subset \mathcal{G}$ of the Lie algebra in the sense that (2.2) and (3.22) induce similar structures in (2.3) and (3.23) respectively. Such a collection $\{S_s\}$, $S = \bigcup_s S_s$, was said in [20] to be resonant with the algebra decomposition $\mathcal{G} = \bigoplus_s V_s$; equation (3.23) was called the resonance condition.

Now, the vector *sub*space of (3.20) $\mathcal{G}'_S \subset \mathcal{G}_S$, defined by

$$\mathcal{G}'_S = \bigoplus_p \left(\bigoplus_{\alpha_p \in S_p} V_{p\alpha_p} \right), \qquad p = 0, \ldots, n, \tag{3.24}$$

(cf. (3.20)) is actually a *subalgebra* (called resonant in [20]) of $\mathcal{G}_S$, $\mathcal{G}'_S \subset \mathcal{G}_S$, with Lie bracket structure given by

$$\mathcal{G}'_S : [V_{p\alpha_p}, V_{q\beta_q}] \subset \bigoplus_{s \le p+q} V_{s\alpha_p\beta_q}, \tag{3.25}$$

and with structure constants determined by (3.18) and the S inner law in (3.19). This is so because the subspace structure (3.25) comes from (2.3) and follows from (3.21), and the r.h.s. of (3.25) is in $\mathcal{G}'_S$ because $\alpha_p\beta_q \in S_s$ $\forall s \le p+q$ due to the resonant condition (3.23).

We now move on to show how the expansions $\mathcal{G}(N_0, \ldots, N_n)$ in Theorem 1 can be retrieved from the above subalgebra $\mathcal{G}'_S$ of $\mathcal{G}_S$. Let us take the following collection of subsets of S

$$S_p = \{\alpha_p \mid \alpha_p = p, \ldots, N+1\}, \quad p = 0, \ldots, n, \tag{3.26}$$

which clearly satisfy (3.23). Let us split them as $S_p = \check{S}_p \cup \hat{S}_p$, $\check{S}_p = \{p, \ldots, N_p\}$, $\hat{S}_p = \{N_p + 1, \ldots, N+1\}$ [20] and use $\hat{S}_p$ to introduce the vector subspace $\hat{\mathcal{G}}'_S \subset \mathcal{G}'_S$ by

$$\hat{\mathcal{G}}'_S = \bigoplus_p \left(\bigoplus_{\alpha_p \in \hat{S}_p} V_{p\alpha_p} \right), \qquad p = 0, \ldots, n \tag{3.27}$$

(c.f. (3.24)). Now, if the integers N_p, $p = 0, \ldots, n$, are chosen to obey the restrictions (3.11) or, equivalently, (3.10), then $\hat{\mathcal{G}}'_S$ is an ideal of $\mathcal{G}'_S$. Indeed, we see from (3.18) that $\hat{\mathcal{G}}'_S$ will be an ideal of $\mathcal{G}'_S$ if for $\alpha_p \in \hat{S}_p$ and $\beta_q \in S_q$ in (3.25), $\alpha_p\beta_q \in \hat{S}_s$ where $s = \min\{n, p+q\}$. This is indeed the case: if $s = p+q \le n$ in (3.25), equation (3.10) for $p \le s$ leads to $(N_p + 1) + q \ge (N_{p+q} + 1)$, and hence $\alpha_p\beta_q \in \hat{S}_{p+q}$. And, if $p+q > n$, $s = n$ in (3.25), equation (3.10) for $p \le n$ gives $N_p \ge N_n - n + p$ and so $N_p + q \ge N_n - n + p + q$. Since now $p + q > n$, this gives $(N_p + 1) + q \ge (N_n + 1)$, and thus $\alpha_p\beta_q \in \hat{S}_n$, $\forall$ $\alpha_p \in \hat{S}_p$, $\forall$ $\beta_q \in S_q$.

The quotient of $\mathcal{G}'_S$ by the ideal $\hat{\mathcal{G}}'_S$,

$$\check{\mathcal{G}}'_S = \mathcal{G}'_S / \hat{\mathcal{G}}'_S \tag{3.28}$$

defines the algebra $\check{\mathcal{G}}'_S$ (not a subalgebra of $\mathcal{G}'_S$), with underlying vector space

$$\check{\mathcal{G}}'_S = \bigoplus_p \left(\bigoplus_{\alpha_p \in \check{S}_p} V_{p\alpha_p} \right), \quad p = 0, \ldots, n. \tag{3.29}$$

As vector spaces, $\hat{\mathcal{G}}'_S$ and $\check{\mathcal{G}}'_S$ are complementary in $\mathcal{G}'_S$. The dimension of $\check{\mathcal{G}}'_S$ is given by

$$\begin{aligned} \dim \check{\mathcal{G}}'_S &= \dim \mathcal{G}'_S - \dim \hat{\mathcal{G}}'_S \\ &= \sum_{p=0}^{n} \sum_{\alpha_p \in \check{S}_p} \dim V_{p\alpha_p} = \sum_{p=0}^{n} \sum_{\alpha_p = p}^{N_p} \dim V_{p\alpha_p} = \sum_{p=0}^{n} (N_p - p + 1) \ \dim V_p. \end{aligned} \tag{3.30}$$

The structure constants of $\check{\mathcal{G}}'_S$ are given by

$$C_{i_p,\beta_p \, j_q,\gamma_q}^{k_s,\alpha_s} = \delta_{\beta_p \gamma_q}^{\alpha_s} c_{i_p j_q}^{k_s}, \quad p, q, s = 0, 1, \ldots, n, \ \alpha_p, \beta_p, \gamma_p = p, \ldots, N_p \tag{3.31}$$

$$= \begin{cases} 0, & \text{if } \beta_p + \gamma_q \neq \alpha_s \\ c_{i_p j_q}^{k_s}, & \text{if } \beta_p + \gamma_q = \alpha_s \end{cases} \quad \begin{array}{l} p, q, s = 0, 1, \ldots, n, \\ i_{p,q,s} = 1, 2, \ldots, \dim V_{p,q,s}, \\ \alpha_p, \beta_p, \gamma_p = p, p+1, \ldots, N_p, \end{array} \tag{3.32}$$

where the part $\delta_{\beta_p \gamma_q}^{\alpha_s}$ of the structure constants (see (3.18)), in which and β_p, γ_q, …, indicate the elements of the subsets $\check{S}_p$, $\check{S}_q$ … above, is obtained from (3.19). We see that the dimensions in (3.30) and (3.8), and the structure constants in (3.32) and (3.6), coincide. Thus, if the integers N_p are restricted as in (3.11), the above algebra $\check{\mathcal{G}}'_S$ is just the expansion $\mathcal{G}(N_0, \ldots, N_n)$ [18] of Theorem 1.

We refer to [20] for further details on S-expanded algebras.

4 The gauge structure of $D = 11$ supergravity

As a recent physical application of the expansion method, we now comment briefly on the underlying gauge structure of eleven-dimensional supergravity [7, 21, 22]. See [17–19, 59–65] for other possible applications of the expansion method.

We are interested here in the underlying gauge symmetry of $D = 11$ Cremmer–Julia–Scherk (CJS) supergravity [6] as a way of understanding the symmetry structure of M-theory, the low energy limit of which is $D = 11$ supergravity. The problem of its hidden or underlying gauge geometry was raised already in the CJS pioneering paper [6], where the possible relevance of $OSp(1|32)$ was suggested. It was specially considered by D'Auria and Fré [7], who looked at the problem as a search for a composite structure of its three-form field $A_3(x)$. Indeed, while two of the $D = 11$ supergravity fields (the graviton $e^a = dx^\mu e^a_\mu(x)$ and the gravitino $\psi^\alpha = dx^\mu \psi^\alpha_\mu(x)$) are given by *one*-form spacetime fields and thus can be considered, together with the spin connection ($\omega^{ab} = dx^\mu \omega^{ab}_\mu(x)$), as gauge fields for the standard super-Poincaré group, the additional $A_{\mu_1\mu_2\mu_3}(x)$ Abelian gauge field in $D = 11$ CJS supergravity is not associated with any super-Poincaré algebra MC *one*-form or generator since it rather corresponds to a *three*-form A_3. However, one may ask whether it is possible to introduce a set of additional one-form fields associated to the LI MC forms of a larger superalgebra such that these fields, together with e^a and ψ^α, can be used to express A_3

in terms of one-forms. If so, the 'old' e^a, ψ^α and the 'new' one-form fields may be considered as gauge fields of a larger supersymmetry group, with A_3 expressed in terms of them. This is what is meant by the underlying gauge group structure of CJS supergravity: it is hidden when the standard $D=11$ supergravity multiplet is considered, and manifest when the three-form field A_3 becomes a *composite* of one-form fields associated with the MC forms of the larger superalgebra, in which case all CJS supergravity fields can be treated as one-form gauge fields. It is then seen that the solution of this problem is equivalent to trivializing a standard $D=11$ supersymmetry algebra $\mathfrak{E}^{(11|32)}$ cohomology four-cocycle ω_4 (structurally equivalent to the four-form dA_3) on a *larger* algebra $\tilde{\mathfrak{E}}$ corresponding to a larger superspace group $\tilde{\Sigma}$.

It turns out [21, 22] that there is a whole one-parameter family of enlarged supersymmetry algebras $\tilde{\mathfrak{E}}(s)$, $s \neq 0$ that trivialize the $\mathfrak{E}^{(11|32)}$ four-cocycle ω_4 ($\sim dA_3$) (see [21, 22] for the meaning of $\tilde{\mathfrak{E}}(s)$ and its associated family of enlarged superspace groups $\tilde{\Sigma}(s)$). Hence (and adding the $D=11$ Lorentz group, $SO(1,10)$), this means that the underlying gauge supergroup of $D=11$ supergravity has a semidirect structure and can be described by any representative of a *one-parametric family of supergroups,* $\tilde{\Sigma}(s) \otimes SO(1,10)$ *for* $s \neq 0$. These may be seen as deformations of $\tilde{\Sigma}(0) \otimes SO(1,10) \subset \tilde{\Sigma}(0) \otimes Sp(32)$, where $\tilde{\Sigma}(0)$ is a certain enlarged superspace group [21, 22]. Thus our conclusion is that the underlying gauge group structure of $D=11$ supergravity is determined by a one-parametric nontrivial deformation of $\tilde{\Sigma}(0) \otimes SO(1,10) \subset \tilde{\Sigma}(0) \otimes Sp(32)$ (two specific cases of the $\tilde{\mathfrak{E}}(s)$ family, $\tilde{\mathfrak{E}}(3/2)$ and $\tilde{\mathfrak{E}}(-1)$, were already found in [7]). The singularity of $\tilde{\mathfrak{E}}(0)$ looks reasonable; the corresponding $\tilde{\Sigma}(0)$ enlarged superspace group is special because the Lorentz $SO(1,10)$ automorphism group of $\tilde{\Sigma}(s)$ ($s \neq 0$) is enhanced to $Sp(32)$ for $\tilde{\Sigma}(0)$. The appearance of $\tilde{\Sigma}(0)$ allows us to clarify the connection of the underlying gauge supergroups with $OSp(1|32)$ above mentioned. It is found [21, 22] that $\tilde{\Sigma}(0) \otimes SO(1,10)$ is an expansion of $OSp(1|32)$; specifically, $\tilde{\Sigma}(0) \otimes SO(1,10) \approx OSp(1|32)(2,3,2)$. It may also be shown that $\tilde{\Sigma}(0) \otimes Sp(32)$ is the expansion of $OSp(1|32)(2,3)$.

The enlarged supersymmetry algebras $\tilde{\mathfrak{E}}(s)$ are central extensions of the M-algebra (of generators $Q_\alpha, P_a, Z_{ab}, Z_{a_1\ldots a_5}$) by an additional fermionic generator Q'_α. Trivializing the $\mathfrak{E}^{(11|32)}$ Lie superalgebra cohomology four-cocycle ω_4 on the enlarged supersymmetry algebra $\tilde{\mathfrak{E}}(s)$, so that ω_4 is the exterior derivative of an invariant form, $\omega_4 = d\tilde{\omega}_3$, is tantamount to finding a composite structure for the three-form field A_3 of CJS supergravity in terms of one-form gauge fields, $A_3 = A_3(e^a, \psi^\alpha; B^{a_1 a_2}, B^{a_1\cdots a_5}, \eta^\alpha)$ associated to the MC forms of $\tilde{\mathfrak{E}}(s)$. The compositeness of A_3 is given by the same equation that provides the $\tilde{\omega}_3$ trivialization $\omega_4 = d\tilde{\omega}_3$ of the ω_4 cocycle (where *now* $\tilde{\omega}_3$ is $\tilde{\Sigma}(s)$-invariant; this is why ω_4 becomes a *trivial* cocycle for $\tilde{\mathfrak{E}}(s)$, $s \neq 0$; see *e.g.* [3]). In the composite A_3 expression, the $\tilde{\mathfrak{E}}(s)$ MC forms are replaced by 'soft' one-forms—spacetime one-form fields—obeying a free differential algebra with curvatures.

The presence of the additional one-form gauge fields associated with the new generators in $\tilde{\mathfrak{E}}(s)$ might be expected. The field $B^{a_1\cdots a_5}$, associated to the $Z_{a_1\ldots a_5}$ M-algebra generator, is needed [66] for a coupling to BPS preons [67, 68], the hypothetical basic constituents of M-theory. In a more conventional perspective, one can notice that the generators $Z_{a_1 a_2}$ and $Z_{a_1\ldots a_5}$ can be treated as topological charges [5] of the M2 and M5 superbranes (see also [69]). In the standard CJS supergravity the M2-brane solution carries a charge of the three-form gauge field A_3 and thus there should have a relation with the charge $Z_{a_1 a_2}$ and its gauge field $B^{a_1 a_2}$. The analysis of the rôle of the fermionic central charge Q'_α and its gauge field η^α in this perspective requires more care, although such a fermionic 'central' charge is also present in the Green algebra [70] (see also [11–14, 71] and references therein).

Some comments are now in order.

• The supergroup manifolds $\tilde{\Sigma}(s)$ are *enlarged* superspaces. The fact that all the *spacetime* fields appearing in the above description of CJS supergravity may be associated to the various coordinates of $\tilde{\Sigma}(s)$ is suggestive of an *enlarged superspace variables/spacetime fields correspondence* principle for $D = 11$ CJS supergravity.

• This is not the only case where such a situation appears. It may be seen [13, 14] that one may introduce an *enlarged superspace variables/worldvolume fields correspondence* principle for superbranes, by which one associates all *worldvolume* fields, including the Born–Infeld (BI) ones [13, 14, 72, 73] in the various D-brane actions, to fields corresponding to forms defined on suitably enlarged superspaces $\tilde{\Sigma}$ (the actual worldvolume fields are the pull-backs of these forms to the worldvolume of the extended supersymmetric object). The worldvolume BI fields, as the spacetime A_3 field of CJS supergravity above, become composite fields. Moreover, a Chevalley–Eilenberg Lie algebra cohomology analysis [13, 14, 74, 75] of the Wess–Zumino terms of many different superbrane actions determines the possible ones and how the ordinary supersymmetry algebra has to be extended (see also [72, 73, 76]). This again suggests an enlarged superspace variables/worldvolume fields correspondence.

• Thus, could there be an *enlarged superspace variables/fields correspondence principle in M-theory*?

To conclude, we would like to mention that the expansion method can also be applied [18] to free differential algebras (FDAs) [7, 77–79], structures that prove useful to discuss the dynamics of supergravity theories. In particular, it can be applied to the *gauge* FDAs obtained by 'softening' the MC forms, and therefore to obtain Chern–Simons type actions, from those for the unexpanded algebras [18, 62, 63] (see [64, 65] for a review of Chern–Simons actions in the supergravity context). The S-expansions [20] briefly reviewed at the end of Sect. 3 can also be applied to the construction of Chern–Simons Lagrangians [80]. The reduction of the $D = 11$ supergravity FDA has been very recently analyzed [81] in terms of the Sezgin algebra [12] and the E_{11} Kac–Moody algebra.

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